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Abstract

We develop a non-anticipative calculus for functionals of a continuous semimartingale, using an extension of the Ito formula to path-dependent functionals which possess certain directional derivatives. The construction is based on a pathwise derivative, introduced by B Dupire, for functionals on the space of right-continuous functions with left limits. We show that this functional derivative admits a suitable extension to the space of square-integrable martingales. This extension defines a weak derivative which is shown to be the inverse of the Ito integral and which may be viewed as a non-anticipative "lifting" of the Malliavin derivative.

These results lead to a constructive martingale representation formula for Ito processes. By contrast with the Clark-Haussmann-Ocone formula, this representation only involves non-anticipative quantities which may be computed pathwise.

Keywords: stochastic calculus, functional calculus, functional Ito formula, Malliavin derivative, martingale representation, semimartingale, Wiener functionals, Clark-Ocone formula.

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Contents

1 Introduction

In the analysis of phenomena with stochastic dynamics, Ito's stochastic calculus [15, 16, 8, 23, 19, 28, 29] has proven to be a powerful and useful tool. A central ingredient of this calculus is the *Ito formula* [15, 16, 23], a change of variable formula for functions $f(X_t)$ of a *semimartingale* X which allows to represent such quantities in terms of a stochastic integral. Given that in many applications such as statistics of processes, physics or mathematical finance, one is led to consider path-dependent functionals of a semimartingale X and its quadratic variation process $[X]$ such as:

$$\int_0^t g(t, X_t)d[X](t), \qquad G(t, X_t, [X]_t), \quad \text{or} \qquad E[G(T, X(T), [X](T))|\mathcal{F}_t] \tag{1}$$

(where $X(t)$ denotes the value at time t and $X_t = (X(u), u \in [0, t])$ the path up to time t) there has been a sustained interest in extending the framework of stochastic calculus to such path-dependent functionals.

In this context, the Malliavin calculus [3, 24, 22, 25, 30, 31, 32] has proven to be a powerful tool for investigating various properties of Brownian functionals. Since the construction of Malliavin derivative does not refer to an underlying filtration $\mathcal{F}_t$, it naturally leads to representations of functionals in terms of *anticipative* processes [4, 14, 25]. However, in most applications it is more natural to consider non-anticipative versions of such representations.

In a recent insightful work, B. Dupire [9] has proposed a method to extend the Ito formula to a functional setting in a *non-anticipative* manner, using a pathwise functional derivative which quantifies the sensitivity of a functional $F_t : D([0,t],\mathbb{R}) \to \mathbb{R}$ to a variation in the endpoint of a path $\omega \in D([0,t],\mathbb{R})$:

$$\nabla_\omega F_t(\omega) = \lim_{\epsilon \to 0} \frac{F_t(\omega + \epsilon 1_t) - F_t(\omega)}{\epsilon}$$

Building on this insight, we develop hereafter a non-anticipative calculus [6] for a class of processes –including the above examples- which may be represented as

$$Y(t) = F_t(\{X(u), 0 \le u \le t\}, \{A(u), 0 \le u \le t\}) = F_t(X_t, A_t) \tag{2}$$

where A is the local quadratic variation defined by $[X](t) = \int_0^t A(u)du$ and the functional

$$F_t : D([0,t],\mathbb{R}^d) \times D([0,t], S_d^+) \to \mathbb{R}$$

represents the dependence of Y on the path $X_t = \{X(u), 0 \le u \le t\}$ of X and its quadratic variation.

Our first result (Theorem 4.1) is a change of variable formula for path-dependent functionals of the form (2). Introducing A_t as additional variable allows us to control the dependence of Y with respect to the "quadratic variation" $[X]$ by requiring smoothness properties of F_t with respect to the variable A_t in the supremum norm, without resorting to p-variation norms as in "rough path" theory [20]. This allows our result to cover a wide range of functionals, including the examples in (1).

We then extend this notion of functional derivative to *processes*: we show that for Y of the form (2) where F satisfies some regularity conditions, the process $\nabla_X Y = \nabla_\omega F(X_t, A_t)$ may be defined intrinsically, independently of the choice of F in (2). The operator ∇_X is shown to admit an extension to the space of square-integrable martingales, which is the inverse of the Ito integral with respect to X: for $\phi \in \mathcal{L}^2(X)$, $\nabla_X \left(\int \phi.dX\right) = \phi$ (Theorem 5.8). In particular, we obtain a constructive version of the martingale representation theorem (Theorem 5.9), which states that for any square-integrable $\mathcal{F}_t^X$-martingale Y,

$$Y(T) = Y(0) + \int_0^T \nabla_X Y.dX \quad \mathbb{P} - a.s.$$

This formula can be seen as a non-anticipative counterpart of the Clark-Haussmann-Ocone formula [4, 13, 14, 18, 25]. The integrand $\nabla_X Y$ is an adapted process which may be computed pathwise, so this formula is more amenable to numerical computations than those based on Malliavin calculus.

Finally, we show that this functional derivative ∇_X may be viewed as a non-anticipative "lifting" of the Malliavin derivative (Theorem 6.1): for square-integrable martingales Y whose terminal values is differentiable in the sense of Malliavin $Y(T) \in \mathbf{D}^{1,2}$, we show that $\nabla_X Y(t) = E[\mathbb{D}_t H|\mathcal{F}_t]$.

These results provide a rigorous mathematical framework for developing and extending the ideas proposed by B. Dupire [9] for a large class of functionals. In particular, unlike the results derived from the pathwise approach viewpoint presented in [5, 9], Theorems 5.8 and 5.9 do not require any pathwise regularity of the functionals and hold for non-anticipative square-integrable processes, including stochastic integrals and functionals which may depend on the quadratic variation of the process.

2 Functional representation of non-anticipative processes

Let $X : [0,T] \times \Omega \mapsto \mathbb{R}^d$ be a continuous, $\mathbb{R}^d-$valued semimartingale defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ assumed to satisfy the usual hypotheses [8]. Denote by $\mathcal{P}$ (resp. $\mathcal{O}$) the associated *predictable* (resp. optional) sigma-algebra on $[0,T]$. $\mathcal{F}_t^X$ denotes the ($\mathbb{P}-$*completed*) natural filtration of X. The paths of X then lie in $C_0([0,T],\mathbb{R}^d)$, which we will view as a subspace of $D([0,t],\mathbb{R}^d)$ the space of cadlag functions with values in $\mathbb{R}^d$. We denote by $[X] = ([X^i, X^j], i,j = 1..d)$ the quadratic (co-)variation process associated to X, taking values in the set S_d^+ of positive $d \times d$ matrices. We assume that

$$[X](t) = \int_0^t A(s)ds \tag{3}$$

for some cadlag process A with values in S_d^+. Note that A need not be a semimartingale. The paths of A lie in $\mathcal{S}_t = D([0,t], S_d^+)$, the space of cadlag functions with values S_d^+.

2.1 Horizontal extension and vertical perturbation of a path

Consider a path $x \in D([0,T]),\mathbb{R}^d)$ and denote by $x_t = (x(u), 0 \le u \le t) \in D([0,t],\mathbb{R}^d)$ its restriction to $[0,t]$ for $t < T$. For a process X we shall similarly denote $X(t)$ its value at t and $X_t = (X(u), 0 \le u \le t)$ its path on $[0,t]$.

For $h \ge 0$, we define the *horizontal* extension $x_{t,h} \in D([0,t+h],\mathbb{R}^d)$ of x_t to $[0,t+h]$ as

$$x_{t,h}(u) = x(u) \qquad u \in [0,t] \ ; \qquad x_{t,h}(u) = x(t) \qquad u \in]t, t+h] \tag{4}$$

For $h \in \mathbb{R}^d$, we define the *vertical* perturbation x_t^h of x_t as the cadlag path obtained by shifting the endpoint by h:

$$x_t^h(u) = x_t(u) \qquad u \in [0,t[\quad x_t^h(t) = x(t) + h \tag{5}$$

or in other words $x_t^h(u) = x_t(u) + h1_{t=u}$.

2.2 Adapted processes as non-anticipative functionals

A process $Y : [0,T] \times \Omega \mapsto \mathbb{R}^d$ adapted to $\mathcal{F}_t^X$ may be represented as

$$Y(t) = F_t(\{X(u), 0 \le u \le t\}, \{A(u), 0 \le u \le t\}) = F_t(X_t, A_t) \tag{6}$$

where $F = (F_t)_{t \in [0,T]}$ is a family of functionals

$$F_t : D([0,t],\mathbb{R}^d) \times \mathcal{S}_t \to \mathbb{R}$$

representing the dependence of $Y(t)$ on the underlying path of X and its quadratic variation.

Since Y is non-anticipative, $Y(t,\omega)$ only depends on the restriction ω_t of ω on $[0,t]$. This motivates the following definition:

Definition 2.1 (Non-anticipative functional)**.** A non-anticipative functional on Υ is a family of functionals $F = (F_t)_{t \in [0,T]}$ where

$$\begin{array}{rcl} F_t : D([0,t],\mathbb{R}^d) \times D([0,t], S_d^+) & \mapsto & \mathbb{R} \\ (x,v) & \to & F_t(x,v) \end{array}$$

is measurable with respect to $\mathcal{B}_t$, the canonical filtration on $D([0,t],\mathbb{R}^d) \times D([0,t], S_d^+)$.

We can also view $F = (F_t)_{t\in[0,T]}$ as a map defined on the space Υ of *stopped paths*:

$$\Upsilon = \{(t, \omega_{t,T-t}), (t,\omega) \in [0,T] \times D([0,T], \mathbb{R}^d \times S_d^+)\} \tag{7}$$

Whenever the context is clear, we will denote a generic element $(t,\omega) \in \Upsilon$ simply by its second component, the path ω stopped at t. Υ can also be identified with the 'vector bundle'

$$\Lambda = \bigcup_{t\in[0,T]} D([0,t], \mathbb{R}^d) \times D([0,t], S_d^+). \tag{8}$$

A natural distance on the space Υ of stopped paths is given by

$$d_\infty((t,\omega),(t',\omega')) = |t-t'| + \sup_{u\in[0,T]} |\omega_{t,T-t}(u) - \omega'_{t',T-t'}(u)| \tag{9}$$

(Υ, d_∞) is then a metric space, a closed subspace of $[0,T]\times D([0,T], \mathbb{R}^d \times S_d^+), \|.\|_\infty)$ for the product topology.

Introducing the process A as additional variable may seem redundant at this stage: indeed $A(t)$ is itself $\mathcal{F}_t-$ measurable i.e. a functional of X_t. However, it is not a *continuous* functional on (Υ, d_∞). Introducing A_t as a second argument in the functional will allow us to control the regularity of Y with respect to $[X]_t = \int_0^t A(u)du$ simply by requiring continuity of F_t in supremum or L^p norms with respect to the "lifted process" (X, A) (see Section 2.3). This idea is analogous in some ways to the approach of rough path theory [20], although here we do not resort to p-variation norms.

If Y is a $\mathcal{B}_t$-predictable process, then [8, Vol. I,Par. 97]

$$\forall t \in [0,T], \qquad Y(t,\omega) = Y(t,\omega_{t-})$$

where ω_{t-} denotes the path defined on $[0,t]$ by

$$\omega_{t-}(u) = \omega(u) \quad u \in [0,t[\qquad \omega_{t-}(t) = \omega(t-)$$

Note that ω_{t-} is cadlag and should *not* be confused with the caglad path $u \mapsto \omega(u-)$.

The functionals discussed in the introduction depend on the process A via $[X] = \int_0^\cdot A(t)dt$. In particular, they satisfy the condition $F_t(X_t, A_t) = F_t(X_t, A_{t-})$. Accordingly, we will assume throughout the paper that all functionals $F_t : D([0,t], \mathbb{R}^d) \times \mathcal{S}_t \to \mathbb{R}$ considered have "predictable" dependence with respect to the second argument:

$$\forall t \in [0,T], \quad \forall (x,v) \in D([0,t], \mathbb{R}^d) \times \mathcal{S}_t, \quad F_t(x_t, v_t) = F_t(x_t, v_{t-}) \tag{10}$$

2.3 Continuity for non-anticipative functionals

We now define a notion of (left) continuity for non-anticipative functionals.

Definition 2.2 (Continuity at fixed times)**.** A functional F defined on Υ is said to be continuous at fixed times for the d_∞ metric if and only if:

$$\forall t \in [0,T), \quad \forall \epsilon > 0, \forall (x,v) \in D([0,t], \mathbb{R}^d) \times \mathcal{S}_t, \quad \exists \eta > 0, (x',v') \in D([0,t], \mathbb{R}^d) \times \mathcal{S}_t,$$
$$d_\infty((x,v),(x',v')) < \eta \Rightarrow |F_t(x,v) - F_t(x',v')| < \epsilon \tag{11}$$

We now define a notion of joint continuity with respect to time and the underlying path:

Definition 2.3 (Continuous functionals)**.** A non-anticipative functional $F = (F_t)_{t\in[0,T)}$ is said to be continuous at $(x, v) \in D([0,t],\mathbb{R}^d) \times \mathcal{S}_t$ if

$$\forall \epsilon > 0, \exists \eta > 0, \forall (x', v') \in \Upsilon, \quad d_\infty((x,v),(x',v')) < \eta \Rightarrow |F_t(x,v) - F_{t'}(x',v')| < \epsilon \tag{12}$$

We denote $\mathbb{C}^{0,0}([0,T))$ the set of non-anticipative functionals continuous on Υ.

Definition 2.4 (Left-continuous functionals)**.** A non-anticipative functional $F = (F_t, t \in [0,T))$ is said to be left-continuous if for each $t \in [0,T)$, $F_t : D([0,t],\mathbb{R}^d) \times \mathcal{S}_t \to \mathbb{R}$ in the sup norm and

$$\forall \epsilon > 0, \forall (x,v) \in D([0,t],\mathbb{R}^d) \times \mathcal{S}_t, \quad \exists \eta > 0, \forall h \in [0,t], \quad \forall (x',v') \in D([0,t-h],\mathbb{R}^d) \times \mathcal{S}_{t-h},$$
$$d_\infty((x,v),(x',v')) < \eta \Rightarrow |F_t(x,v) - F_{t-h}(x',v')| < \epsilon \tag{13}$$

We denote $\mathbb{C}^{0,0}_l([0,T))$ the set of left-continuous functionals.

We define analogously the class of right continuous functionals $\mathbb{C}^{0,0}_r([0,T))$.
We call a functional "boundedness preserving" if it is bounded on each bounded set of paths:

Definition 2.5 (Boundedness-preserving functionals)**.** Define $\mathbb{B}([0,T))$ as the set of non-anticipative functionals F such that for every compact subset K of $\mathbb{R}^d$, every $R > 0$ and $t_0 < T$:

$$\exists C_{K,R,t_0} > 0, \quad \forall t \le t_0, \forall (x,v) \in D([0,t],K) \times \mathcal{S}_t, \sup_{s\in[0,t]} |v(s)| < R \Rightarrow |F_t(x,v)| < C_{K,R,t_0} \tag{14}$$

2.4 Measurability properties

Composing a non-anticipative functional F with the process (X, A) yields an $\mathcal{F}_t-$adapted process $Y(t) = F_t(X_t, A_t)$. The results below link the measurability and pathwise regularity of Y to the regularity of the functional F.

Lemma 2.6 (Pathwise regularity)**.** *If $F \in \mathbb{C}^{0,0}_l$ then for any $(x,v) \in D([0,T],\mathbb{R}^d) \times \mathcal{S}_T$, the path $t \mapsto F_t(x_{t-}, v_{t-})$ is left-continuous.*

Proof. Let $F \in \mathbb{C}^{0,0}_l$ and $t \in [0,T)$. For $h > 0$ sufficiently small,

$$d_\infty((x_{t-h}, v_{t-h}), (x_{t-}, v_{t-})) = \sup_{u\in(t-h,t)} |x(u) - x(t-h)| + \sup_{u\in(t-h,t)} |v(u) - v(t-h)| + h \tag{15}$$

Since x and v are cadlag, this quantity converges to 0 as $h \to 0+$, so

$$F_{t-h}(x_{t-h}, v_{t-h}) - F_t(x_{t-}, v_{t-}) \overset{h\to 0^+}{\to} 0$$

so $t \mapsto F_t(x_{t-}, v_{t-})$ is left-continuous. □

Theorem 2.7. *(i) If F is continuous at fixed times, then the process Y defined by $Y((x,v),t) = F_t(x_t, v_t)$ is adapted.*

(ii) If $F \in \mathbb{C}^{0,0}_l([0,T))$, then the process $Z(t) = F_t(X_t, A_t)$ is optional.

(iii) If $F \in \mathbb{C}_l^{0,0}([0,T))$, and if either A is continuous or F verifies (10)*, then Z is a predictable process.*

In particular, any $F \in \mathbb{C}_l^{0,0}$ is a non-anticipative functional in the sense of Definition 2.1. We propose an easy-to-read proof of points (i) and (iii) in the case where A is continuous. The (more technical) proof for the cadlag case is given in the Appendix A.

Continuous case. Assume that F is continuous at fixed times and that the paths of (X, A) are almost-surely continuous. Let us prove that Y is $\mathcal{F}_t$−adapted: $X(t)$ is $\mathcal{F}_t$-measurable. Introduce the partition $t_n^i = \frac{iT}{2^n}, i = 0..2^n$ of $[0,T]$, as well as the following piecewise-constant approximations of X and A:

$$X^n(t) = \sum_{k=0}^{2^n} X(t_k^n) 1_{[t_k^n, t_{k+1}^n)}(t) + X_T 1_{\{T\}}(t)$$
$$A^n(t) = \sum_{k=0}^{2^n} A(t_k^n) 1_{[t_k^n, t_{k+1}^n)}(t) + A_T 1_{\{T\}}(t) \tag{16}$$

The random variable $Y^n(t) = F_t(X_t^n, A_t^n)$ is a continuous function of the random variables $\{X(t_k^n), A(t_k^n), t_k^n \le t\}$ hence is $\mathcal{F}_t$-measurable. The representation above shows in fact that $Y^n(t)$ is $\mathcal{F}_t$-measurable. X_t^n and A_t^n converge respectively to X_t and A_t almost-surely so $Y^n(t) \to^{n\to\infty} Y(t)$ a.s., hence $Y(t)$ is $\mathcal{F}_t$-measurable.
(i) implies point (iii) since the path of Z are left-continuous by Lemma 2.6. □

3 Pathwise derivatives of non-anticipative functionals

3.1 Horizontal and vertical derivatives

We now define pathwise derivatives for a functional $F = (F_t)_{t\in[0,T)} \in \mathbb{C}^{0,0}$, following Dupire [9].

Definition 3.1 (Horizontal derivative)**.** The *horizontal derivative* at $(x, v) \in D([0,t], \mathbb{R}^d) \times \mathcal{S}_t$ of non-anticipative functional $F = (F_t)_{t\in[0,T)}$ is defined as

$$\mathcal{D}_t F(x, v) = \lim_{h\to 0^+} \frac{F_{t+h}(x_{t,h}, v_{t,h}) - F_t(x_t, v_t)}{h} \tag{17}$$

if the corresponding limit exists. If (17) is defined for all $(x, v) \in \Upsilon$ the map

$$\begin{array}{rcl} \mathcal{D}_t F : D([0,t], \mathbb{R}^d) \times \mathcal{S}_t & \mapsto & \mathbb{R}^d \\ (x, v) & \to & \mathcal{D}_t F(x, v) \end{array} \tag{18}$$

defines a non-anticipative functional $\mathcal{D}F = (\mathcal{D}_t F)_{t\in[0,T]}$, the *horizontal derivative* of F.

Note that our definition (17) is different from the one in [9] where the case $F(x, v) = G(x)$ is considered.

Dupire [9] also introduced a pathwise spatial derivative for such functionals, which we now introduce. Denote $(e_i, i = 1..d)$ the canonical basis in $\mathbb{R}^d$.

Definition 3.2. A non-anticipative functional $F = (F_t)_{t\in[0,T)}$ is said to be *vertically differentiable* at $(x, v) \in D([0, t]), \mathbb{R}^d) \times D([0, t], S_d^+)$ if

$$\begin{array}{rcl} \mathbb{R}^d & \mapsto & \mathbb{R} \\ e & \to & F_t(x_t^e, v_t) \end{array}$$

is differentiable at 0. Its gradient at 0

$$\nabla_x F_t\ (x, v) = (\partial_i F_t(x, v),\ i = 1..d) \qquad \text{where} \quad \partial_i F_t(x, v) = \lim_{h\to 0} \frac{F_t(x_t^{he_i}, v) - F_t(x, v)}{h} \tag{19}$$

is called the *vertical derivative* of F_t at (x, v). If (19) is defined for all $(x, v) \in \Upsilon$, the maps

$$\begin{array}{rcl} \nabla_x F : D([0, t], \mathbb{R}^d) \times \mathcal{S}_t & \mapsto & \mathbb{R}^d \\ (x, v) & \to & \nabla_x F_t(x, v) \end{array} \tag{20}$$

define a non-anticipative functional $\nabla_x F = (\nabla_x F_t)_{t\in[0,T]}$, the *vertical derivative* of F. F is then said to be *vertically differentiable* on Υ.

Remark 3.3. $\partial_i F_t(x, v)$ is simply the directional derivative of F_t in direction $(1_{\{t\}} e_i, 0)$. Note that this involves examining cadlag perturbations of the path x, even if x is continuous.

Remark 3.4. If $F_t(x, v) = f(t, x(t))$ with $f \in C^{1,1}([0, T) \times \mathbb{R}^d)$ then we retrieve the usual partial derivatives:

$$\mathcal{D}_t F(x, v) = \partial_t f(t, X(t)) \qquad \nabla_x F_t(X_t, A_t) = \nabla_x f(t, X(t)).$$

Remark 3.5. Bismut [3] considered directional derivatives of functionals on $D([0, T], \mathbb{R}^d)$ in the direction of purely discontinuous (e.g. piecewise constant) functions with finite variation, which is similar to Def. 3.2. This notion, used in [3] to derive an integration by parts formula for pure-jump processes, is natural in the context of discontinuous semimartingales. We will show that the directional derivative (19) also intervenes naturally when the underlying process X is *continuous*, which is less obvious.

Definition 3.6 (Regular functionals)**.** Define $\mathbb{C}^{1,k}([0, T))$ as the set of functionals $F \in \mathbb{C}_l^{0,0}$ which are

- horizontally differentiable with $\mathcal{D}_t F$ continuous at fixed times,
- k times vertically differentiable with $\nabla_x^j F \in \mathbb{C}_l^{0,0}([0, T))$ for $j = 1..k$.

Define $\mathbb{C}_b^{1,k}([0, T))$ as the set of functionals $F \in \mathbb{C}^{1,2}$ such that $\mathcal{D}F, \nabla_x F, ..., \nabla_x^k F \in \mathbb{B}([0, T))$.

We denote $\mathbb{C}^{1,\infty}([0, T)) = \cap_{k\geq 1}\mathbb{C}^{1,k}([0, T)$.

Note that this notion of regularity only involves directional derivatives with respect to *local* perturbations of paths, so $\nabla_x F$ and $\mathcal{D}_t F$ seems to contain *less* information on the behavior of F than, say, the Fréchet derivative which consider perturbations in all directions in $C_0([0, T], \mathbb{R}^d)$ or the Malliavin derivative [21, 22] which examines perturbations in the direction of all absolutely continuous functions. Nevertheless we will show in Section 4 that knowledge of $\mathcal{D}F, \nabla_x F, \nabla_x^2 F$ along the paths of X derivatives are sufficient to reconstitute the path of $Y(t) = F_t(X_t, A_t)$.

Example 1 (Smooth functions). In the case where F reduces to a smooth *function* of $X(t)$,

$$F_t(x_t, v_t) = f(t, x(t)) \tag{21}$$

where $f \in C^{1,k}([0,T] \times \mathbb{R}^d)$, the pathwise derivatives reduces to the usual ones: $F \in \mathbb{C}_b^{1,k}$ with:

$$\mathcal{D}_t F(x_t, v_t) = \partial_t f(t, x(t)) \qquad \nabla_x^j F_t(x_t, v_t) = \partial_x^j f(t, x(t)) \tag{22}$$

In fact to have $F \in \mathbb{C}^{1,k}$ we just need f to be right-differentiable in the time variable, with right-derivative $\partial_t f(t,.)$ which is continuous in the space variable and $f, \nabla f$ and $\nabla^2 f$ to be jointly left-continuous in t and continuous in the space variable.

Example 2 (Cylindrical functionals). Let $g \in C^0(\mathbb{R}^d, \mathbb{R}), h \in C^k(\mathbb{R}^d, \mathbb{R})$ with $h(0) = 0$. Then

$$F_t(\omega) = h\left(\omega(t) - \omega(t_n-)\right) \quad 1_{t \geq t_n} \quad g(\omega(t_1-), \omega(t_2-)..., \omega(t_n-))$$

is in $\mathbb{C}_b^{1,k}$ with $\mathcal{D}_t F(\omega) = 0$ and

$$\forall j = 1..k, \quad \nabla_\omega^j F_t(\omega) = h^{(j)}\left(\omega(t) - \omega(t_n-)\right) \quad 1_{t \geq t_n} g\left(\omega(t_1-), \omega(t_2-)..., \omega(t_n-)\right)$$

Example 3 (Integrals with respect to quadratic variation). A process $Y(t) = \int_0^t g(X(u))d[X](u)$ where $g \in C_0(\mathbb{R}^d)$ may be represented by the functional

$$F_t(x_t, v_t) = \int_0^t g(x(u))v(u)du \tag{23}$$

It is readily observed that $F \in \mathbb{C}_b^{1,\infty}$, with:

$$\mathcal{D}_t F(x_t, v_t) = g(x(t))v(t) \qquad \nabla_x^j F_t(x_t, v_t) = 0 \tag{24}$$

Example 4. The martingale $Y(t) = X(t)^2 - [X](t)$ is represented by the functional

$$F_t(x_t, v_t) = x(t)^2 - \int_0^t v(u)du \tag{25}$$

Then $F \in \mathbb{C}_b^{1,\infty}$ with:

$$\begin{aligned} \mathcal{D}_t F(x, v) = -v(t) \qquad & \nabla_x F_t(x_t, v_t) = 2x(t) \\ \nabla_x^2 F_t(x_t, v_t) = 2 \qquad & \nabla_x^j F_t(x_t, v_t) = 0, j \geq 3 \end{aligned} \tag{26}$$

Example 5. $Y = \exp(X - [X]/2)$ may be represented as $Y(t) = F(X_t)$

$$F_t(x_t, v_t) = e^{x(t) - \frac{1}{2}\int_0^t v(u)du} \tag{27}$$

Elementary computations show that $F \in \mathbb{C}_b^{1,\infty}$ with:

$$\mathcal{D}_t F(x, v) = -\frac{1}{2}v(t)F_t(x, v) \qquad \nabla_x^j F_t(x_t, v_t) = F_t(x_t, v_t) \tag{28}$$

Note that, although A_t may be expressed as a functional of X_t, this functional is not continuous and without introducing the second variable $v \in \mathcal{S}_t$, it is not possible to represent Examples 3, 4 and 5 as a left-continuous functional of x alone.

3.2 Obstructions to regularity

It is instructive to observe what prevents a functional from being regular in the sense of Definition 3.6. The examples below illustrate the fundamental obstructions to regularity:

Example 6 (Delayed functionals). Let $\epsilon > 0$. $F_t(x_t, v_t) = x(t - \epsilon)$ defines a $\mathbb{C}_b^{0,\infty}$ functional. All vertical derivatives are 0. However, F fails to be horizontally differentiable.

Example 7 (Jump of x at the current time). $F_t(x_t, v_t) = x(t) - x(t-)$ defines a functional which is infinitely differentiable and has regular pathwise derivatives:

$$\mathcal{D}_t F(x_t, v_t) = 0 \qquad \nabla_x F_t(x_t, v_t) = 1 \tag{29}$$

However, the functional itself fails to be $\mathbb{C}_l^{0,0}$.

Example 8 (Jump of x at a fixed time). $F_t(x_t, v_t) = 1_{t \geq t_0}(x(t_0) - x(t_0-))$ defines a functional in $\mathbb{C}_l^{0,0}$ which admits horizontal and vertical derivatives at any order at each point (x, v). However, $\nabla_x F_t(x_t, v_t) = 1_{t=t_0}$ fails to be either right- or left-continuous so F is not $\mathbb{C}^{0,1}$ in the sense of Definition 3.2.

Example 9 (Maximum). $F_t(x_t, v_t) = \sup_{s \leq t} x(s)$ is $\mathbb{C}_l^{0,0}$ but fails to be vertically differentiable on the set

$$\{(x_t, v_t) \in D([0,t], \mathbb{R}^d) \times \mathcal{S}_t, \quad x(t) = \sup_{s \leq t} x(s)\}.$$

4 Functional Ito calculus

4.1 Functional Ito formula

We are now ready to prove our first main result, which is a change of variable formula for non-anticipative functionals of a semimartingale [6, 9]:

Theorem 4.1. *For any non-anticipative functional $F \in \mathbb{C}_b^{1,2}$ verifying (10) and any $t \in [0, T)$,*

$$\begin{aligned} F_t(X_t, A_t) - F_0(X_0, A_0) &= \int_0^t \mathcal{D}_u F(X_u, A_u) du + \int_0^t \nabla_x F_u(X_u, A_u).dX(u) \\ &+ \int_0^t \frac{1}{2} \mathrm{tr}\left(\nabla_x^2 F_u(X_u, A_u)\; d[X](u)\right) \qquad a.s. \end{aligned} \tag{30}$$

In particular, for any $F \in \mathbb{C}_b^{1,2}$, $Y(t) = F_t(X_t, A_t)$ is a semimartingale.

(30) shows that, for a regular functional $F \in \mathbb{C}^{1,2}([0,T))$, the process $Y = F(X, A)$ may be reconstructed from the second-order jet $(\mathcal{D}F, \nabla_x F, \nabla_x^2 F)$ of F along the paths of X.

Proof. Let us first assume that X does not exit a compact set K and that $\|A\|_\infty \leq R$ for some $R > 0$. Let us introduce a sequence of random partitions $(\tau_k^n, k = 0..k(n))$ of $[0, t]$, by adding the jump times of A to the dyadic partition $(t_i^n = \frac{it}{2^n}, i = 0..2^n)$:

$$\tau_0^n = 0 \qquad \tau_k^n = \inf\{s > \tau_{k-1}^n | 2^n s \in \mathbb{N} \text{ or } \quad |A(s) - A(s-)| > \frac{1}{n}\} \wedge t \tag{31}$$

The following arguments apply pathwise. Lemma A.3 ensures that

$$\eta_n = \sup\{|A(u) - A(\tau_i^n)| + |X(u) - X(\tau_i^n)| + \frac{t}{2^n}, i \le 2^n, u \in [\tau_i^n, \tau_{i+1}^n)\} \underset{n\to\infty}{\to} 0.$$

Denote ${}_nX = \sum_{i=0}^{\infty} X(\tau_{i+1}^n) 1_{[\tau_i^n, \tau_{i+1}^n)} + X(t) 1_{\{t\}}$ which is a cadlag piecewise constant approximation of X_t, and ${}_nA = \sum_{i=0}^{\infty} A(\tau_i^n) 1_{[\tau_i^n, \tau_{i+1}^n)} + A(t) 1_{\{t\}}$ which is an adapted cadlag piecewise constant approximation of A_t. Denote $h_i^n = \tau_{i+1}^n - \tau_i^n$. Start with the decomposition:

$$\begin{aligned} F_{\tau_{i+1}^n}({}_nX_{\tau_{i+1}^n-}, {}_nA_{\tau_{i+1}^n-}) - F_{\tau_i^n}({}_nX_{\tau_i^n-}, {}_nA_{\tau_i^n-}) &= F_{\tau_{i+1}^n}({}_nX_{\tau_{i+1}^n-}, {}_nA_{\tau_i^n, h_i^n}) - F_{\tau_i^n}({}_nX_{\tau_i^n}, {}_nA_{\tau_i^n}) \\ &+ F_{\tau_i^n}({}_nX_{\tau_i^n}, {}_nA_{\tau_i^n-}) - F_{\tau_i^n}({}_nX_{\tau_i^n-}, {}_nA_{\tau_i^n-}) \end{aligned} \tag{32}$$

where we have used the fact that F has predictable dependence in the second variable to have $F_{\tau_i^n}({}_nX_{\tau_i^n}, {}_nA_{\tau_i^n}) = F_{\tau_i^n}({}_nX_{\tau_i^n}, {}_nA_{\tau_i^n-})$. The first term in (32) can be written $\psi(h_i^n) - \psi(0)$ where:

$$\psi(u) = F_{\tau_i^n+u}({}_nX_{\tau_i^n,u}, {}_nA_{\tau_i^n,u}) \tag{33}$$

Since $F \in \mathbb{C}^{1,2}([0,T])$, ψ is right-differentiable and left-continuous by Lemma 2.6, so:

$$F_{\tau_{i+1}^n}({}_nX_{\tau_i^n, h_i^n}, {}_nA_{\tau_i^n, h_i^n}) - F_{\tau_i^n}({}_nX_{\tau_i^n}, {}_nA_{\tau_i^n}) = \int_0^{\tau_{i+1}^n - \tau_i^n} \mathcal{D}_{\tau_i^n+u} F({}_nX_{\tau_i^n,u}, {}_nA_{\tau_i^n,u}) du \tag{34}$$

The second term in (32) can be written $\phi(X(\tau_{i+1}^n) - X(\tau_i^n)) - \phi(0)$ where $\phi(u) = F_{\tau_i^n}({}_nX^u_{\tau_i^n-}, {}_nA_{\tau_i^n})$. Since $F \in \mathbb{C}_b^{1,2}$, ϕ is a C^2 function and $\phi'(u) = \nabla_x F_{\tau_i^n}({}_nX^u_{\tau_i^n-}, {}_nA_{\tau_i^n, h_i})$, $\phi''(u) = \nabla_x^2 F_{\tau_i^n}({}_nX^u_{\tau_i^n-}, {}_nA_{\tau_i^n, h_i})$. Applying the Ito formula to ϕ between 0 and $\tau_{i+1}^n - \tau_i^n$ and the $(\mathcal{F}_{\tau_i+s})_{s\ge 0}$ continuous semimartingale $(X(\tau_i^n + s))_{s\ge 0}$, yields:

$$\begin{aligned} \phi(X(\tau_{i+1}^n) - X(\tau_i^n)\ ) - \phi(0) &= \int_{\tau_i^n}^{\tau_{i+1}^n} \nabla_x F_{\tau_i^n}({}_nX^{X(s)-X(\tau_i^n)}_{\tau_i^n-}, {}_nA_{\tau_i^n}) dX(s) \\ &+ \frac{1}{2} \int_{\tau_i^n}^{\tau_{i+1}^n} \mathrm{tr}\left[{}^t\nabla_x^2 F_{\tau_i^n}({}_nX^{X(s)-X(\tau_i^n)}_{\tau_i^n-}, {}_nA_{\tau_i^n}) d[X](s)\right] \end{aligned} \tag{35}$$

Summing over $i \ge 0$ and denoting $i(s)$ the index such that $s \in [\tau_{i(s)}^n, \tau_{i(s)+1}^n)$, we have shown:

$$\begin{aligned} F_t({}_nX_t, {}_nA_t) - F_0(X_0, A_0) &= \int_0^t \mathcal{D}_s F({}_nX_{\tau_{i(s)}^n, s-\tau_{i(s)}^n}, {}_nA_{\tau_{i(s)}^n, s-\tau_{i(s)}^n}) ds \\ &+ \int_0^t \nabla_x F_{\tau_{i(s)+1}^n}({}_nX^{X(s)-X(\tau_{i(s)}^n)}_{\tau_{i(s)}^n-}, {}_nA_{\tau_{i(s)}^n, h_{i(s)}}) dX(s) \\ &+ \frac{1}{2} \int_0^t \mathrm{tr}\left[\nabla_x^2 F_{\tau_{i(s)}^n}({}_nX^{X(s)-X(\tau_{i(s)}^n)}_{\tau_{i(s)}^n-}, {}_nA_{\tau_{i(s)}^n}).d[X](s)\right] \end{aligned} \tag{36}$$

$F_t({}_nX_t, {}_nA_t)$ converges to $F_t(X_t, A_t)$ almost surely. Since all approximations of (X, A) appearing in the various integrals have a d_∞-distance from (X_s, A_s) less than $\eta_n \to 0$, the continuity at fixed times of $\mathcal{D}F$ and left-continuity $\nabla_x F$, $\nabla_x^2 F$ imply that the integrands appearing in the above integrals converge respectively to $\mathcal{D}_s F(X_s, A_s)$, $\nabla_x F_s(X_s, A_s)$, $\nabla_x^2 F_s(X_s, A_s)$ as $n \to \infty$. Since the derivatives are in $\mathbb{B}$ the integrands in the various above integrals are bounded by a constant dependant only on F,K and R and t does not depend on s nor on ω. The dominated convergence and the dominated convergence theorem for the stochastic integrals [28, Ch.IV Theorem 32] then ensure that the Lebesgue-Stieltjes integrals converge almost surely, and the stochastic integral in probability, to the terms appearing in (30) as $n \to \infty$.

Consider now the general case where X and A may be unbounded. Let K_n be an increasing sequence of compact sets with $\bigcup_{n\geq 0} K_n = \mathbb{R}^d$ and denote the optional stopping times

$$\tau_n = \inf\{s < t | X_s \notin K^n \text{ or } |A_s| > n\} \wedge t.$$

Applying the previous result to the stopped process $(X_{t\wedge\tau_n}, A_{t\wedge\tau_n})$ and noting that, by (10), $F_t(X_t, A_t) = F_t(X_t, A_{t-})$ leads to:

$$F_t(X_{t\wedge\tau_n}, A_{t\wedge\tau_n}) - F_0(Z_0, A_0) = \int_0^{t\wedge\tau_n} \mathcal{D}_u F_u(X_u, A_u) du + \frac{1}{2}\int_0^{t\wedge\tau_n} \mathrm{tr}\left({}^t\nabla_x^2 F_u(X_u, A_u) d[X](u)\right)$$
$$+ \int_0^{t\wedge\tau_n} \nabla_x F_u(X_u, A_u).dX + \int_{t\wedge\tau^n}^t D_u F(X_{u\wedge\tau_n}, A_{u\wedge\tau_n}) du$$

The terms in the first line converges almost surely to the integral up to time t since $t \wedge \tau_n = t$ almost surely for n sufficiently large. For the same reason the last term converges almost surely to 0. □

Remark 4.2. The above proof is probabilistic and makes use of the (classical) Ito formula [15]. In the companion paper [5] we give a non-probabilistic proof of Theorem 4.1, using the analytical approach of Föllmer [12], which allows X to have discontinuous (cadlag) trajectories.

Example 10. If $F_t(x_t, v_t) = f(t, x(t))$ where $f \in C^{1,2}([0,T] \times \mathbb{R}^d)$, (30) reduces to the standard Itô formula.

Example 11. For the functional in Example 5) $F_t(x_t, v_t) = e^{x(t) - \frac{1}{2}\int_0^t v(u)du}$, the formula (30) yields the well-known integral representation

$$\exp(X(t) - \frac{1}{2}[X](t)\ ) = \int_0^t e^{X(u) - \frac{1}{2}[X](u)} dX(u) \tag{37}$$

An immediate corollary of Theorem 4.1 is that, if X is a local martingale, any $\mathbb{C}_b^{1,2}$ functional of X which has finite variation is equal to the integral of its horizontal derivative:

Corollary 4.3. *If X is a local martingale and $F \in \mathbb{C}_b^{1,2}$, the process $Y(t) = F_t(X_t, A_t)$ has finite variation if only if $\nabla_x F_t(X_t, A_t) = 0$ $d[X] \times d\mathbb{P}$-almost everywhere.*

Proof. $Y(t)$ is a continuous semimartingale by Theorem 4.1, with semimartingale decomposition given by (30). If Y has finite variation, then by formula (30), its continuous martingale component should be zero i.e. $\int_0^t \nabla_x F_t(X_t, A_t).dX(t) = 0$ a.s. Computing its quadratic variation, we obtain

$$\int_0^T \mathrm{tr}\left({}^t\nabla_x F_t(X_t, A_t).\nabla_x F_t(X_t, A_t).d[X]\right) = 0$$

which implies in particular that $\|\partial_i F_t(X_t, A_t)\|^2 = 0$ $d[X^i] \times d\mathbb{P}$-almost everywhere for $i = 1..d$. Thus, $\nabla_x F_t(X_t, A_t) = 0$ for $(t, \omega) \notin A \subset [0,T] \times \Omega$ where $\int_A d[X^i] \times d\mathbb{P} = 0$ for $i = 1..d$. □

4.2 Vertical derivative of an adapted process

For a ($\mathcal{F}_t-$adapted) process Y, the the functional representation (42) is not unique, and the vertical $\nabla_x F$ depends on the choice of representation F. However, Theorem 4.1 implies that the *process* $\nabla_x F_t(X_t, A_t)$ has an intrinsic character i.e. independent of the chosen representation:

Corollary 4.4. *Let* $F^1, F^2 \in \mathbb{C}_b^{1,2}([0,T)\ )$*, such that:*

$$\forall t \in [0,T), \quad F_t^1(X_t, A_t) = F_t^2(X_t, A_t) \qquad \mathbb{P} - a.s. \tag{38}$$

Then, outside an evanescent set:

$$^t[\nabla_x F_t^1(X_t, A_t) - \nabla_x F_t^2(X_t, A_t)]A(t-)[\nabla_x F_t^1(X_t, A_t) - \nabla_x F_t^2(X_t, A_t)] = 0 \tag{39}$$

Proof. Let $X(t) = B(t) + M(t)$ where B is a continuous process with finite variation and M is a continuous local martingale. There exists $\Omega_1 \subset \Omega$ such that $\mathbb{P}(\Omega_1) = 1$ and for $\omega \in \Omega$ the path of $t \mapsto X(t, \omega)$ is continuous and $t \mapsto A(t, \omega)$ is cadlag. Theorem 4.1 implies that the local martingale part of $0 = F^1(X_t, A_t) - F^2(X_t, A_t)$ can be written:

$$0 = \int_0^t \left[\nabla_x F_u^1(X_u, A_u) - \nabla_x F_u^2(X_u, A_u)\right] dM(u) \tag{40}$$

Considering its quadratic variation, we have, on Ω_1

$$0 = \int_0^t \frac{1}{2}\,^t[\nabla_x F_u^1(X_u, A_u) - \nabla_x F_u^2(X_u, A_u)]A(u-)[\nabla_x F_u^1(X_u, A_u) - \nabla_x F_u^2(X_u, A_u)]du \tag{41}$$

By Lemma 2.6 ($\nabla_x F^1(X_t, A_t) = \nabla_x F^1(X_{t-}, A_{t-})$ since X is continuous and F verifies (10). So on Ω_1 the integrand in (41) is left-continuous; therefore (41) implies that for $t < T$ and $\omega \in \Omega_1$,

$$^t[\nabla_x F_u^1(X_u, A_u) - \nabla_x F_u^2(X_u, A_u)]A(u-)[\nabla_x F_u^1(X_u, A_u) - \nabla_x F_u^2(X_u, A_u) = 0$$

. $\square$

In the case where for all $t < T$, $A(t-)$ is almost surely positive definite, Corollary 4.4 allows to define intrinsically the pathwise derivative of a process Y which admits a functional representation $Y(t) = F_t(X_t, A_t)$:

Definition 4.5 (Vertical derivative of a process)**.** Define $\mathcal{C}_b^{1,2}(X)$ the set of $\mathcal{F}_t$-adapted processes Y which admit a functional representation in $\mathbb{C}_b^{1,2}$:

$$\mathcal{C}_b^{1,2}(X) = \{Y, \quad \exists F \in \mathbb{C}_b^{1,2} \quad Y(t) = F_t(X_t, A_t) \quad \mathbb{P} - \text{ a.s.}\} \tag{42}$$

If $A(t)$ is non-singular i.e. $det(A(t)) \neq 0$ $dt \times d\mathbb{P}$ almost-everywhere then for any $Y \in \mathcal{C}_b^{1,2}(X)$, the predictable process:

$$\nabla_X Y(t) = \nabla_x F_t(X_t, A_t)$$

is uniquely defined up to an evanescent set, independently of the choice of $F \in \mathbb{C}_b^{1,2}$ in the representation (42). We will call $\nabla_X Y$ the *vertical derivative* of Y with respect to X.

In particular this construction applies to the case where X is a standard Brownian motion, where $A = I_d$, so we obtain the existence of a vertical derivative process for $\mathbb{C}_b^{1,2}$ Brownian functionals:

Definition 4.6 (Vertical derivative of non-anticipative Brownian functionals)**.** Let W be a standard d-dimensional Brownian motion. For any $Y \in \mathcal{C}_b^{1,2}(W)$ with representation $Y(t) = F_t(W_t, t)$, the predictable process

$$\nabla_W Y(t) = \nabla_x F_t(W_t, t)$$

is uniquely defined up to an evanescent set, independently of the choice of $F \in \mathbb{C}_b^{1,2}$.

5 Martingale representation formulas

Consider now the case where X is a Brownian martingale:

Assumption 5.1. $X(t) = X(0) + \int_0^t \sigma(u).dW(u)$ where σ is a process adapted to $\mathcal{F}_t^W$ verifying

$$\det(\sigma(t)) \neq 0 \quad dt \times d\mathbb{P} - a.e. \tag{43}$$

The functional Ito formula (Theorem 4.1) then leads to an explicit martingale representation formula for $\mathcal{F}_t$-martingales in $\mathcal{C}_b^{1,2}(X)$. This result may be seen as a non-anticipative counterpart of the Clark-Haussmann-Ocone formula [4, 25, 14] and generalizes other constructive martingale representation formulas previously obtained using Markovian functionals [7, 10, 11, 17, 26], Malliavin calculus [2, 18, 14, 25, 24] or other techniques [1, 27].

Consider an $\mathcal{F}_T$ measurable random variable H with $E|H| < \infty$ and consider the martingale $Y(t) = E[H|\mathcal{F}_t]$.

5.1 A martingale representation formula

If Y admits a representation $Y(t) = F_t(X_t, A_t)$ where $F \in \mathbb{C}_b^{1,2}$, we obtain the following stochastic integral representation for Y in terms of its derivative $\nabla_X Y$ with respect to X:

Theorem 5.2. *If $Y(t) = F_t(X_t, A_t)$ for some functional $F \in \mathbb{C}_b^{1,2}$, then:*

$$Y(T) = Y(0) + \int_0^T \nabla_x F_t(X_t, A_t) dX(t) = Y(0) + \int_0^T \nabla_X Y.dX \tag{44}$$

Note that regularity assumptions are not on $H = Y(T)$ but on the functionals $Y(t) = E[H|\mathcal{F}_t], t < T$, which is typically more regular than H itself.

Proof. Theorem 4.1 implies that for $t \in [0, T)$:

$$\begin{aligned} Y(t) = [\int_0^t \mathcal{D}_u F(X_u, A_u) du + \frac{1}{2} \int_0^t \mathrm{tr}[{}^t\nabla_x^2 F_u(X_u, A_u) d[X](u)] \\ + \int_0^t \nabla_x F_u(X_u, A_u) dX(u) \end{aligned} \tag{45}$$

Given the regularity assumptions on F, the first term in this sum is a continuous process with finite variation while the second is a continuous local martingale. However, Y is a martingale and its decomposition as sum of a finite variation process and a local martingale is unique [29]. Hence the first term is 0 and: $Y(t) = \int_0^t F_u(X_u, A_u)dX_u$. Since $F \in \mathbb{C}_l^{0,0}([0,T])$ $Y(t)$ has limit $F_T(X_T, A_T)$ as $t \to T$, so the stochastic integral also converges. □

Example 12.

If $e^{X(t)-\frac{1}{2}[X](t)}$ is a martingale, applying Theorem 5.2 to the functional $F_t(x_t, v_t) = e^{x(t)-\int_0^t v(u)du}$ yields the familiar formula:

$$e^{X(t)-\frac{1}{2}[X](t)} = 1 + \int_0^t e^{X(s)-\frac{1}{2}[X](s)}dX(s) \tag{46}$$

5.2 Extension to square-integrable functionals

Let $\mathcal{L}^2(X)$ be the Hilbert space of progressively-measurable processes ϕ such that:

$$||\phi||^2_{\mathcal{L}^2(X)} = E\left[\int_0^t \phi_s^2 d[X](s)\right] < \infty \tag{47}$$

and $\mathcal{I}^2(X)$ be the space of square-integrable stochastic integrals with respect to X:

$$\mathcal{I}^2(X) = \{\int_0^\cdot \phi(t)dX(t), \phi \in \mathcal{L}^2(X)\} \tag{48}$$

endowed with the norm $||Y||_2^2 = E[Y(T)^2]$ The Ito integral $I_X : \phi \mapsto \int_0^\cdot \phi_s dX(s)$ is then a bijective isometry from $\mathcal{L}^2(X)$ to $\mathcal{I}^2(X)$.

We will now show that the operator $\nabla_X :\mapsto \mathcal{L}^2(X)$ admits a suitable extension to $\mathcal{I}^2(X)$ which verifies

$$\forall \phi \in \mathcal{L}^2(X), \qquad \nabla_X\left(\int \phi.dX\right) = \phi, \qquad dt \times d\mathbb{P} - a.s. \tag{49}$$

i.e. ∇_X is the inverse of the Ito stochastic integral with respect to X.

Definition 5.3 (Space of test processes)**.** The space of *test processes* $D(X)$ is defined as

$$D(X) = \mathcal{C}_b^{1,2}(X) \cap \mathcal{I}^2(X) \tag{50}$$

Theorem 5.2 allows to define intrinsically the vertical derivative of a process in $D(X)$ as an element of $\mathcal{L}^2(X)$.

Definition 5.4. Let $Y \in D(X)$, define the process $\nabla_X Y \in \mathcal{L}^2(X)$ as the equivalence class of $\nabla_x F_t(X_t, A_t)$, which does not depend on the choice of the representation functional $Y(t) = F_t(X_t, A_t)$

Proposition 5.5 (Integration by parts on $D(X)$)**.** *Let $Y, Z \in D(X)$. Then:*

$$E\left[Y(T)Z(T)\right] = E\left[\int_0^T \nabla_X Y(t)\nabla_X Z(t)d[X](t)\right] \tag{51}$$

Proof. Let $Y, Z \in D(X) \subset \mathcal{C}_b^{1,2}(X)$. Then Y, Z are martingales with $Y(0) = Z(0) = 0$ and $E[|Y(T)|^2] < \infty, E[|Z(T)|^2] < \infty$. Applying Theorem 5.2 to Y and Z, we obtain

$$E\left[Y(T)Z(T)\right] = E[\int_0^T \nabla_X Y dX \quad \int_0^T \nabla_X Z dX]$$

Applying the Ito isometry formula yields the result. □

Using this result, we can extend the operator ∇_X in a weak sense to a suitable space of the space of (square-integrable) stochastic integrals, where $\nabla_X Y$ is characterized by (51) being satisfied against all test processes.

The following definition introduces the Hilbert space $\mathcal{W}^{1,2}(X)$ of martingales on which ∇_X acts as a weak derivative, characterized by integration-by-part formula (51). This definition may be also viewed as a non-anticipative counterpart of Wiener-Sobolev spaces in the Malliavin calculus [22, 30].

Definition 5.6 (Martingale Sobolev space)**.** The Martingale Sobolev space $\mathcal{W}^{1,2}(X)$ is defined as the closure in $\mathcal{I}^2(X)$ of $D(X)$.

The Martingale Sobolev space $\mathcal{W}^{1,2}(X)$ is in fact none other than $\mathcal{I}^2(X)$, the set of square-integrable stochastic integrals:

Lemma 5.7. $\{\nabla_X Y, Y \in D(X)\}$ *is dense in* $\mathcal{L}^2(X)$ *and*

$$\mathcal{W}^{1,2}(X) = \mathcal{I}^2(X).$$

Proof. We first observe that the set U of "cylindrical" processes of the form

$$\phi_{n,f,(t_1,..,t_n)}(t) = f(X(t_1), ..., X(t_n))1_{t>t_n}$$

where $n \geq 1$, $0 \leq t_1 < .. < t_n \leq T$ and $f \in C_b^\infty(\mathbb{R}^n, \mathbb{R})$ is a total set in $\mathcal{L}^2(X)$ i.e. the linear span of U is dense in $\mathcal{L}^2(X)$. For such an integrand $\phi_{n,f,(t1,..,t_n)}$, the stochastic integral with respect to X is given by the martingale

$$Y(t) = I_X(\phi_{n,f,(t_1,..,t_n)})(t) = F_t(X_t, A_t)$$

where the functional F is defined on Υ as:

$$F_t(x_t, v_t) = f(x(t_1-), ..., x(t_n-))(x(t) - x(t_n))1_{t>t_n}$$

so that:

$$\nabla_x F_t(x_t, v_t) = f(x_{t_1-}, ..., x_{t_n-})1_{t>t_n}, \nabla_x^2 F_t(x_t, v_t) = 0, \mathcal{D}_t F(x_t, v_t) = 0$$

which shows that $F \in \mathbb{C}_b^{1,2}$ (see Example 2). Hence, $Y \in \mathcal{C}_b^{1,2}(X)$. Since f is bounded, Y is obviously square integrable so $Y \in D(X)$. Hence $I_X(U) \subset D(X)$.

Since I_X is a bijective isometry from $\mathcal{L}^2(X)$ to $\mathcal{I}^2(X)$, the density of U in $\mathcal{L}^2(X)$ entails the density of $I_X(U)$ in $\mathcal{I}^2(X)$, so $\mathcal{W}^{1,2}(X) = \mathcal{I}^2(X)$. □

Theorem 5.8 (Extension of ∇_X to $\mathcal{W}^{1,2}(X)$)**.** *The vertical derivative $\nabla_X : D(X) \mapsto \mathcal{L}^2(X)$ is closable on $\mathcal{W}^{1,2}(X)$. Its closure defines a bijective isometry*

$$\begin{aligned} \nabla_X : \quad \mathcal{W}^{1,2}(X) &\mapsto \mathcal{L}^2(X) \\ \int_0^{\cdot} \phi.dX &\mapsto \phi \end{aligned} \tag{52}$$

characterized by the following integration by parts formula*: for $Y \in \mathcal{W}^{1,2}(X)$, $\nabla_X Y$ is the unique element of $\mathcal{L}^2(X)$ such that*

$$\forall Z \in D(X), \qquad E[Y(T)Z(T)] = E\left[\int_0^T \nabla_X Y(t) \nabla_X Z(t) d[X](t)\right]. \tag{53}$$

In particular, ∇_X is the adjoint of the Ito stochastic integral

$$\begin{aligned} I_X : \mathcal{L}^2(X) &\mapsto \mathcal{W}^{1,2}(X) \\ \phi &\mapsto \int_0^{\cdot} \phi.dX \end{aligned} \tag{54}$$

in the following sense:

$$\forall \phi \in \mathcal{L}^2(X), \quad \forall Y \in \mathcal{W}^{1,2}(X), \quad E[Y(T) \int_0^T \phi.dX] = E[\int_0^T \nabla_X Y \;\phi d[X] \quad] \tag{55}$$

Proof. Any $Y \in \mathcal{W}^{1,2}(X)$ may be written as $Y(t) = \int_0^t \phi(s) dX(s)$ with $\phi \in \mathcal{L}^2(X)$, which is uniquely defined $d[X] \times d\mathbb{P}$ a.e. The Ito isometry formula then guarantees that (53) holds for ϕ. To show that (53) uniquely characterizes ϕ, consider $\psi \in \mathcal{L}^2(X)$ which also satisfies (53), then, denoting $I_X(\psi) = \int_0^{\cdot} \psi dX$ its stochastic integral with respect to X, (53) then implies that

$$\forall Z \in D(X), \quad < I_X(\psi) - Y, Z >_{\mathcal{W}^{1,2}(X)} = E[(Y(T) - \int_0^T \psi dX) Z(T)] = 0$$

which implies $I_X(\psi) = Y \; d[X] \times d\mathbb{P}$ a.e. since by construction $D(X)$ is dense in $\mathcal{W}^{1,2}(X)$. Hence, $\nabla_X : D(X) \mapsto \mathcal{L}^2(X)$ is closable on $\mathcal{W}^{1,2}(X)$.

This construction shows that $\nabla_X : \mathcal{W}^{1,2}(X) \mapsto \mathcal{L}^2(X)$ is a bijective isometry which coincides with the adjoint of the Ito integral on $\mathcal{W}^{1,2}(X)$. $\square$

Thus, the Ito integral I_X with respect to X

$$I_X : \mathcal{L}^2(X) \mapsto \mathcal{W}^{1,2}(X)$$

admits an inverse on $\mathcal{W}^{1,2}(X)$ which is an extension of the (pathwise) vertical derivative ∇_X operator introduced in Definition 3.2, and

$$\forall \phi \in \mathcal{L}^2(X), \qquad \nabla_X \left(\int_0^{\cdot} \phi dX \right) = \phi \tag{56}$$

holds in the sense of equality in $\mathcal{L}^2(X)$.

The above results now allow us to state a general version of the martingale representation formula, valid for all square-integrable martingales:

Theorem 5.9 (Martingale representation formula: general case)**.** *For any square-integrable $\mathcal{F}_t^X$-martingale Y,*

$$Y(T) = Y(0) + \int_0^T \nabla_X Y dX \quad \mathbb{P} - a.s.$$

6 Relation with the Malliavin derivative

The above results hold in particular in the case where $X = W$ is a Brownian motion. In this case, the vertical derivative ∇_W may be related to the *Malliavin derivative* [22, 2, 3, 31] as follows.

Consider the canonical Wiener space $(\Omega_0 = C_0([0,T], \mathbb{R}^d), \|.\|_\infty, \mathbb{P})$ endowed with its Borelian σ-algebra, the filtration of the canonical process. Consider an $\mathcal{F}_T$-measurable functional $H = H(X(t), t \in [0,T]) = H(X_T)$ with $E[|H|^2] < \infty$. If H is differentiable in the Malliavin sense [2, 22, 24, 31] e.g. $H \in \mathbf{D}^{1,2}$ with Malliavin derivative $\mathbb{D}_t H$, then the Clark-Haussmann-Ocone formula [25, 24] gives a stochastic integral representation of H in terms of the Malliavin derivative of H:

$$H = E[H] + \int_0^T {}^p E[\mathbb{D}_t H | \mathcal{F}_t] dW_t \tag{57}$$

where ${}^p E[\mathbb{D}_t H | \mathcal{F}_t]$ denotes the predictable projection of the Malliavin derivative. This yields a stochastic integral representation of the martingale $Y(t) = E[H|\mathcal{F}_t]$:

$$Y(t) = E[H|\mathcal{F}_t] = E[H] + \int_0^t {}^p E[\mathbb{D}_t H | \mathcal{F}_u] dW_u$$

Related martingale representations have been obtained under a variety of conditions [2, 7, 11, 18, 26, 24].

Denote by

- $L^2([0,T] \times \Omega)$ the set of (anticipative) processes ϕ on $[0,T]$ with $E\int_0^T \|\phi(t)\|^2 dt < \infty$.
- $\mathbb{D}$ the Malliavin derivative operator, which associates to a random variable $H \in \mathbf{D}^{1,2}(0,T)$ the (anticipative) process $(\mathbb{D}_t H)_{t \in [0,T]} \in L^2([0,T] \times \Omega)$.

Theorem 6.1 (Lifting theorem)**.** *The following diagram is commutative is the sense of $dt \times d\mathbb{P}$ equality:*

$$\begin{array}{ccc}
\mathcal{I}^2(W) & \overset{\nabla_W}{\rightarrow} & \mathcal{L}^2(W) \\
\uparrow (E[.|\mathcal{F}_t])_{t\in[0,T]} & & \uparrow (E[.|\mathcal{F}_t])_{t\in[0,T]} \\
\mathbf{D}^{1,2} & \overset{\mathbb{D}}{\rightarrow} & L^2([0,T]\times\Omega)
\end{array}$$

In other words, the conditional expectation operator intertwines ∇_W with the Malliavin derivative:

$$\forall H \in L^2(\Omega_0, \mathcal{F}_T, \mathbb{P}), \qquad \nabla_W \left(E[H|\mathcal{F}_t]\right) = E[\mathbb{D}_t H | \mathcal{F}_t] \tag{58}$$

Proof. The Clark-Haussmann-Ocone formula [25] gives

$$\forall H \in \mathbf{D}^{1,2}, \qquad H = E[H] + \int_0^T {}^p E[\mathbb{D}_t H | \mathcal{F}_t] dW_t \tag{59}$$

where ${}^{p}E[\mathbb{D}_t H|\mathcal{F}_t]$ denotes the predictable projection of the Malliavin derivative. On other hand theorem 5.2 gives:

$$\forall H \in L^2(\Omega_0, \mathcal{F}_T, \mathbb{P}), \qquad H = E[H] + \int_0^T \nabla_W Y(t)\ dW(t) \tag{60}$$

where $Y(t) = E[H|\mathcal{F}_t]$. Hence ${}^{p}E[\mathbb{D}_t H|\mathcal{F}_t] = \nabla_W E[H|\mathcal{F}_t]$, $dt \times d\mathbb{P}$ almost everywhere. □

Thus, the conditional expectation operator (more precisely: the *predictable* projection on $\mathcal{F}_t$ [8, Vol. I]) can be viewed as a morphism which "lifts" relations obtained in the framework of Malliavin calculus into relations between non-anticipative quantities, where the Malliavin derivative and the Skorokhod integral are replaced, respectively, by the vertical derivative ∇_W and the Ito stochastic integral.

From a computational viewpoint, unlike the Clark-Haussmann-Ocone representation which requires to simulate the *anticipative* process $\mathbb{D}_t H$ and compute conditional expectations, $\nabla_X Y$ only involves non-anticipative quantities which can be computed path by path. It is thus more amenable to numerical computations. This topic is further explored in a forthcoming work.

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A Proof of Theorem 2.7

In order to prove theorem 2.7 in the general case where A is only required to be cadlag, we need the following three lemmas. The first lemma states a property analogous to 'uniform continuity' for cadlag functions:

Lemma A.1. *Let f be a cadlag function on $[0,T]$ and define $\Delta f(t) = f(t) - f(t-)$. Then*

$$\forall \epsilon > 0, \quad \exists \eta(\epsilon) > 0, \quad |x-y| \leq \eta \Rightarrow |f(x) - f(y)| \leq \epsilon + \sup_{t \in (x,y]} \{|\Delta f(t)|\} \tag{61}$$

Proof. If (61) does not hold, then there exists a sequence $(x_n, y_n)_{n\geq 1}$ such that $x_n \leq y_n$, $y_n - x_n \to 0$ but $|f(x_n) - f(y_n)| > \epsilon + \sup_{t\in[x_n,y_n]}\{|\Delta f(t)|\}$. We can extract a convergent subsequence $(x_{\psi(n)})$ such that $x_{\psi(n)} \to x$. Noting that either an infinity of terms of the sequence are less than x or an infinity are more than x, we can extract *monotone* subsequences $(u_n, v_n)_{n\geq 1}$ which converge to x. If $(u_n), (v_n)$ both converge to x from above or from below, $|f(u_n) - f(v_n)| \to 0$ which yields a contradiction. If one converges from above and the other from below, $\sup_{t\in[u_n,v_n]}\{|\Delta f(t)|\} \geq |\Delta f(x)|$ but $|f(u_n) - f(v_n)| \to |\Delta f(x)|$, which results in a contradiction as well. Therefore (61) must hold. □

Lemma A.2. *If $\alpha \in \mathbb{R}$ and V is an adapted cadlag process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and σ is a optional time, then:*

$$\tau = \inf\{t > \sigma, \quad |V(t) - V(t-)| > \alpha\} \tag{62}$$

is a stopping time.

Proof. We can write that:

$$\{\tau \leq t\} = \bigcup_{q \in \mathbb{Q} \bigcap [0,t)} (\{\sigma \leq t - q\} \bigcap \{ \sup_{t \in (t-q,t]} |V(u) - V(u-)| > \alpha\} \tag{63}$$

and, using Lemma A.1,

$$\{ \sup_{u\in(t-q,t]} |V(u) - V(u-)| > \alpha\} = \bigcup_{n_0 > 1} \bigcap_{n > n_0} \bigcup_{m \geq 1} \{ \sup_{1 \leq i \leq 2^n} |V(t - q\frac{i-1}{2^n}) - V(t - q\frac{i}{2^n})| > \alpha + \frac{1}{m}\}. \tag{64}$$

□

Lemma A.3 (Uniform approximation of cadlag functions by step functions)**.**
Let $f \in D([0,T],\mathbb{R}^d)$ and $\pi^n = (t_i^n)_{n\geq 1, i=0..k_n}$ a sequence of partitions $(0 = t_0^n < t_1 < ... < t_{k_n}^n = T)$ of $[0,T]$ such that:

$$\sup_{0\leq i\leq k_n-1} |t_{i+1}^n - t_i^n| \overset{n\to\infty}{\to} 0 \qquad \sup_{u\in[0,T]\setminus\pi^n} |\Delta f(u)| \overset{n\to\infty}{\to} 0$$

$$\text{then} \qquad \sup_{u\in[0,T]} |f(u) - \sum_{i=0}^{k_n-1} f(t_i^n) 1_{[t_i^n, t_{i+1}^n)}(u) + f(t_{k_n}^n) 1_{\{t_{k_n}^n\}}(u)| \overset{n\to\infty}{\to} 0 \tag{65}$$

Proof. Denote $h^n = f - \sum_{i=0}^{k_n-1} f(t_i^n) 1_{[t_i^n, t_{i+1}^n)} + f(t_{k_n}^n) 1_{\{t_{k_n}^n\}}$. Since $f - h^n$ is piecewise constant on π^n and $h^n(t_i^n) = 0$ by definition,

$$\sup_{t\in[0,T]} |h^n(t)| = \sup_{i=0..k_n-1} \sup_{[t_i^n, t_i^{n+1})} |h^n(t)| = \sup_{t_i^n < t < t_i^{n+1}} |f(t) - f(t_i^n)|$$

Let $\epsilon > 0$. For $n \geq N$ sufficiently large, $\sup_{u\in[0,T]\setminus\pi^n} |\Delta f(u)| \leq \epsilon/2$ and $\sup_i |t_{i+1}^n - t_i^n| \leq \eta(\epsilon/2)$ using the notation of Lemma A.1. Then, applying Lemma A.1 to f we obtain, for $n \geq N$,

$$\sup_{t\in[t_i^n, t_i^{n+1})} |f(t) - f(t_i^n)| \leq \frac{\epsilon}{2} + \sup_{t_i^n < t < t_i^{n+1}} |\Delta f(u)| \leq \epsilon.$$

$\square$

We can now prove Theorem 2.7 in the case where A is a cadlag adapted process.
Proof of Theorem 2.7: Let us first show that $F_t(X_t, A_t)$ is adapted. Define:

$$\tau_0^N = 0 \quad \tau_k^N = \inf\{t > \tau_{k-1}^N | 2^N t \in \mathbb{N} \text{ or } \quad |A(t) - A(t-)| > \frac{1}{N}\} \wedge t \tag{66}$$

From lemma A.2, τ_k^N are stopping times. Define the following piecewise constant approximations of X_t and A_t along the partition $(\tau_k^N, k \geq 0)$:

$$\begin{aligned} X^N(s) &= \sum_{k\geq 0} X_{\tau_k^N} 1_{[\tau_k^N, \tau_{k+1}^N[}(s) + X(t) 1_{\{t\}}(s) \\ A^N(s) &= \sum_{k=0} A_{\tau_k^N} 1_{[\tau_k^N, \tau_{k+1}^N)}(t) + A(t) 1_{\{t\}}(s) \end{aligned} \tag{67}$$

as well as their truncations of rank K:

$$_K X^N(s) = \sum_{k=0}^K X_{\tau_k^N} 1_{[\tau_k^N, \tau_{k+1}^N)}(s) \qquad _K A^N(t) = \sum_{k=0}^K A_{\tau_k^N} 1_{[\tau_k^N, \tau_{k+1}^N)}(t) \tag{68}$$

Since $(_K X_t^N, _K A_t^N)$ coincides with (X_t^N, A_t^N) for K sufficiently large,

$$F_t(X_t^N, A_t^N) = \lim_{K\to\infty} F_t(_K X_t^N, _K A_t^N). \tag{69}$$

The approximations $F_t^n(_K X_t^N, _K A_t^N)$ are $\mathcal{F}_t$-measurable as they are continuous functions of the random variables:

$$\{(X(\tau_k^N) 1_{\tau_k^N \leq t}, A(\tau_k^N) 1_{\tau_k^N \leq t}), k \leq K\}$$

so their limit $F_t(X_t^N, A_t^N)$ is also $\mathcal{F}_t$-measurable. Thanks to Lemma A.3, X_t^N and A_t^N converge uniformly to X_t and A_t, hence $F_t(X_t^N, A_t^N)$ converges to $F_t(X_t, A_t)$ since $F_t : (D([0,t], \mathbb{R}^d) \times \mathcal{S}_t, \|.\|_\infty) \to \mathbb{R}$ is continuous.

To show the optionality of Z in point (ii), we will show that Z it as limit of right-continuous adapted processes. For $t \in [0,T]$, define $i^n(t)$ to be the integer such that $t \in [\frac{iT}{n}, \frac{(i+1)T}{n})$. Define the process: $Z_t^n = F_{\frac{(i^n(t))T}{n}}(X_{\frac{(i^n(t))T}{n}}, A_{\frac{(i^n(t))T}{n}})$, which is piecewise-constant and has right-continuous trajectories, and is also adapted by the first part of the theorem. Since $F \in \mathbb{C}_l^{0,0}$, $Z^n(t) \to Z(t)$ almost surely, which proves that Z is optional. Point (iii) follows from (i) and lemma 2.6, since in both cases $F_t(X_t, A_t) = F_t(X_{t-}, A_{t-})$ hence Z has left-continuous trajectories.